Imposition of natural and essential boundary conditions in embedded meshless methods using nodal integration

Gabriel Fougeron[1,2] Guillaume Pierrot[1] Denis Aubry[2]

[1]ESI Group, Rungis, France

[2]CentraleSupélec, Châtenay-Malabry, France

CentraleSupélec get it right®

Diffusion equation : Continuous weak formulation

Find $u \in H^1(\Omega)$ such that :

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v = \int_\Omega sv + \int_{\partial\Omega_N} gv \quad \forall v \in H^1_{0,D}(\Omega) \\ u_{|\partial\Omega_D} = u_0 \end{cases}$$

Operator Framework

Diffusion equation : Continuous weak formulation

Find $u \in H^1(\Omega)$ such that :

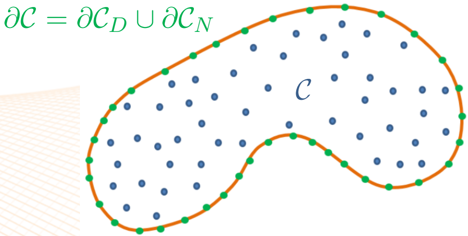
$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v = \int_\Omega sv + \int_{\partial\Omega_N} gv \quad \forall v \in H^1_{0,D}(\Omega) \\ u_{|\partial\Omega_D} = u_0 \end{cases}$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} \displaystyle\int_{\mathcal{C}} \nabla u \cdot \nabla v = \int_{\mathcal{C}} sv + \int_{\partial\mathcal{C}_N} gv \quad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial\mathcal{C}_D} = u_0 \end{cases}$$

CentraleSupélec esi get it right®



Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} \displaystyle\int_{\mathcal{C}} \nabla u \cdot \nabla v = \int_{\mathcal{C}} sv + \int_{\partial\mathcal{C}_N} gv \qquad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial\mathcal{C}_D} = u_0 \end{cases}$$

Operator Framework

$$
\begin{aligned}
H^1(\Omega) \quad &\longrightarrow \quad H^1(\mathcal{C}) = (\mathcal{C} \to \mathbb{R}) \\
H^1_{0,D}(\Omega) \quad &\longrightarrow \quad H^1_{0,D}(\mathcal{C}) = (\mathcal{C} \backslash \partial\mathcal{C}_D \to \mathbb{R})
\end{aligned}
$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$
\begin{cases}
\int_{\mathcal{C}} \nabla u \cdot \nabla v = \int_{\mathcal{C}} sv + \int_{\partial\mathcal{C}_N} gv \qquad \forall v \in H^1_{0,D}(\mathcal{C}) \\
u_{|\partial\mathcal{C}_D} = u_0
\end{cases}
$$

Operator Framework

$$\int_{\Omega} \longrightarrow \int\!\!\!\!\!\square_{\mathcal{C}} \quad \text{positive, linear}$$

$$\overset{\text{def}}{\Rightarrow} V_i > 0 \quad \forall i \in \mathcal{C}$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} \int\!\!\!\!\!\square_{\mathcal{C}} \nabla u \cdot \nabla v = \int\!\!\!\!\!\square_{\mathcal{C}} sv + \int\!\!\!\!\!\square_{\partial\mathcal{C}_N} gv \quad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial\mathcal{C}_D} = u_0 \end{cases}$$

CentraleSupélec get it right®

$$\int_\Omega \longrightarrow ⨋_{\mathcal{C}} \quad \text{positive, linear}$$

$$\overset{\text{def}}{\Rightarrow} V_i > 0 \quad \forall i \in \mathcal{C}$$

$$\int_\Omega f \mathrm{d}V \sim ⨋_{\mathcal{C}} f = \sum_{i \in \mathcal{C}} V_i f(\mathbf{x}_i)$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} ⨋_{\mathcal{C}} \nabla u \cdot \nabla v = ⨋_{\mathcal{C}} sv + \oint_{\partial \mathcal{C}_N} gv \quad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial \mathcal{C}_D} = u_0 \end{cases}$$

CentraleSupélec   esi get it right®

$$\int_{\partial\Omega} \longrightarrow \oint_{\partial\mathcal{C}} \quad \text{positive, linear}$$

$$\overset{\text{def}}{\Rightarrow} \Gamma_i > 0 \quad \forall i \in \partial\mathcal{C}$$

$$\oint_{\partial\mathcal{C}} f = \sum_{i\in\mathcal{C}} \Gamma_i f_i$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} \oint_{\mathcal{C}} \nabla u \cdot \nabla v = \oint_{\mathcal{C}} sv + \oint_{\partial\mathcal{C}_N} gv \quad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial\mathcal{C}_D} = u_0 \end{cases}$$

CentraleSupélec get it right®

$$\nabla \quad \longrightarrow \quad \mathbb{V} \quad \text{linear}$$

$$V_i \mathbb{V}_i f = \sum_{j \in \mathcal{C}} \mathbf{A}_{i,j} f_j$$

Discrete weak formulation

Find $u \in H^1(\mathcal{C})$ such that :

$$\begin{cases} \int_{\mathcal{C}} \nabla u \cdot \nabla v = \int_{\mathcal{C}} sv + \int_{\partial \mathcal{C}_N} gv & \forall v \in H^1_{0,D}(\mathcal{C}) \\ u_{|\partial \mathcal{C}_D} = u_0 & \end{cases}$$

Constraints on point cloud generation . . .

. . . in the interior . . .

- "Harmonious" point clouds ⇒ Lower consistency error
- Case well-covered in the litterature
  - [Löhner, R., & Onate, E. (1998)]
  - [Fattal, R. (2011)]
  - [De Goes, F. & Desbrun, M. (2012)]
- very efficient solutions exist

Constraints on point cloud generation . . .

CentraleSupélec get it right®

. . . in the interior . . .

- "Harmonious" point clouds $\Rightarrow$ Lower consistency error
- Case well-covered in the litterature
  - [Löhner, R., & Onate, E. (1998)]
  - [Fattal, R. (2011)]
  - [De Goes, F. & Desbrun, M. (2012)]
- very efficient solutions exist

. . . and on the boundary

- Added constraint on nodal positions : $\mathbf{x} \in \partial\Omega$
- Seldom covered in the litterature

. . . in the interior . . .

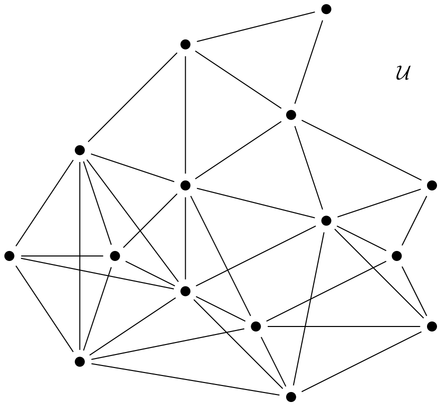
- "Harmonious" point clouds $\Rightarrow$ Lower consistency error
- Case well-covered in the litterature
  - [Löhner, R., & Onate, E. (1998)]
  - [Fattal, R. (2011)]
  - [De Goes, F. & Desbrun, M. (2012)]
- very efficient solutions exist

. . . and on the boundary

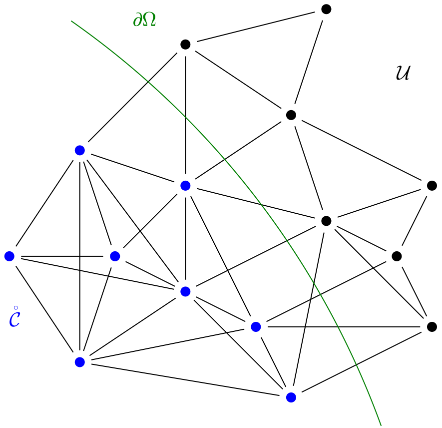
- Added constraint on nodal positions : $\mathbf{x} \in \partial\Omega$
- Seldom covered in the litterature

Completely bypass the generation of a boundary fitted cloud and design an embedded meshless method ?

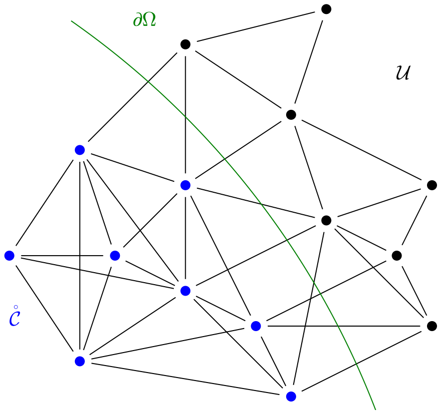
Embedding a point cloud in the geometry



Embedding a point cloud in the geometry

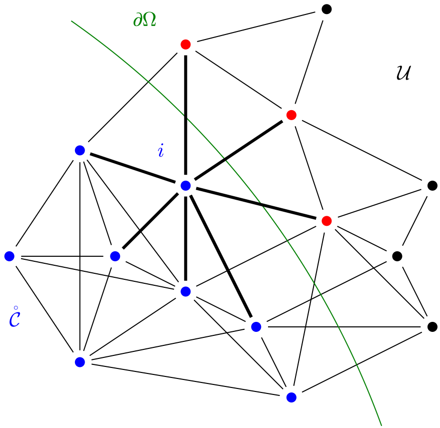


Embedding a point cloud in the geometry

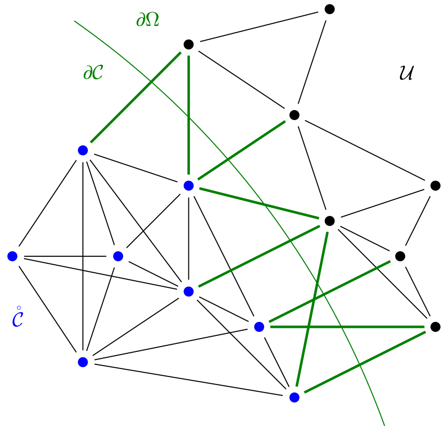


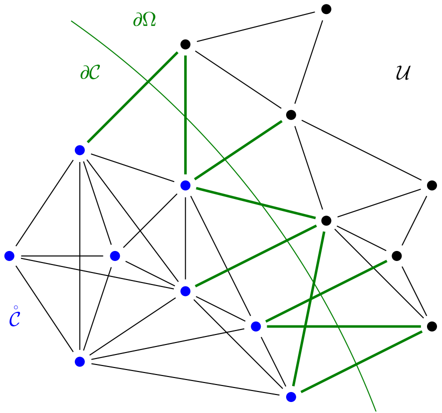
$$\oint_{\mathcal{C}} f \stackrel{\text{def}}{=} \oint_{\mathcal{U}} f \delta_{\mathring{\mathcal{C}}} = \sum_{i \in \mathring{\mathcal{C}}} V_i f_i$$

Embedding a point cloud in the geometry



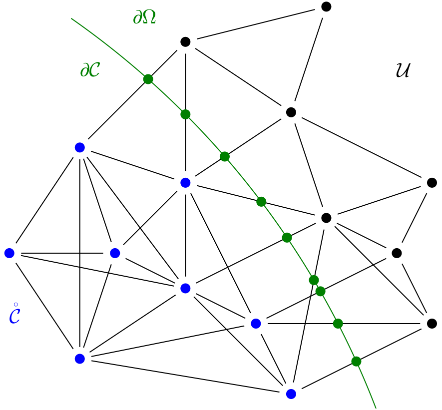
Embedding a point cloud in the geometry



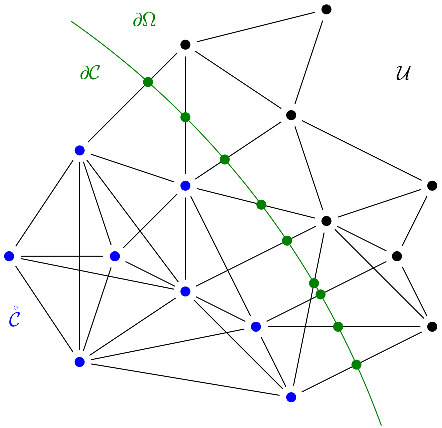


$$\begin{cases} \mathbf{x}_b = (1-\alpha_b)\mathbf{x}_i + \alpha_b \mathbf{x}_o \in \partial\Omega \\ \alpha_b \in [0,1] \end{cases}$$

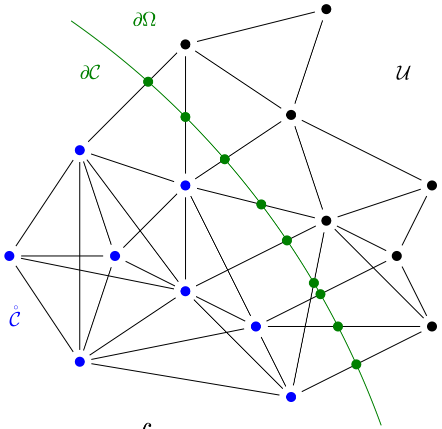
Embedding a point cloud in the geometry



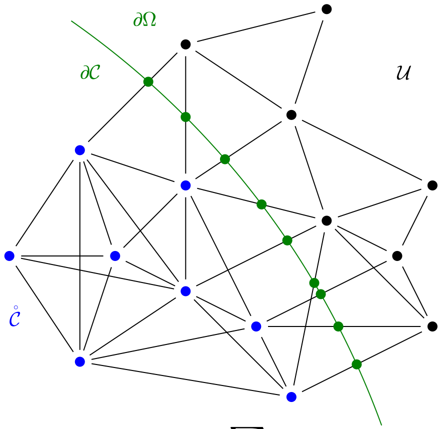
$$\begin{cases} \mathbf{x}_b = (1-\alpha_b)\mathbf{x}_i + \alpha_b \mathbf{x}_o \in \partial\Omega \\ \alpha_b \in [0,1] \end{cases}$$



$$\mathcal{C} = \mathring{\mathcal{C}} \cup \partial\mathcal{C}$$

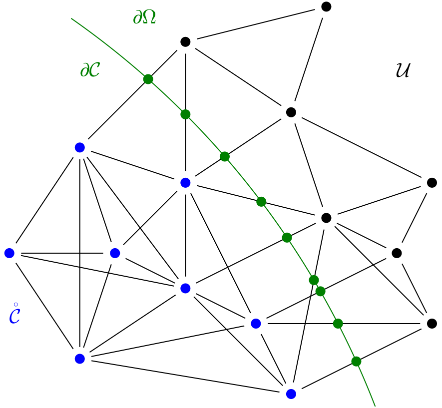


$$\oint_{\partial\mathcal{C}} f = \sum_{b\in\partial\mathcal{C}} \Gamma_b f_b$$

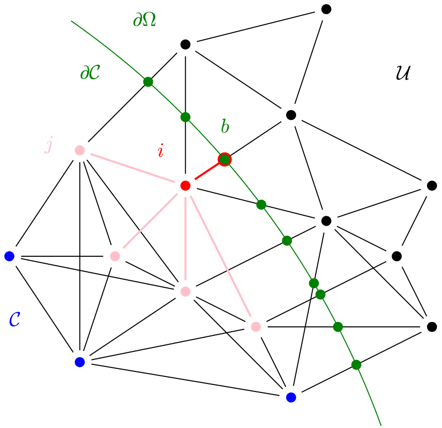


$$V_i \mathbb{V}_i f = \sum_{j \in \mathring{\mathcal{C}}} \mathbf{A}_{i,j} f_j$$

Embedding a point cloud in the geometry

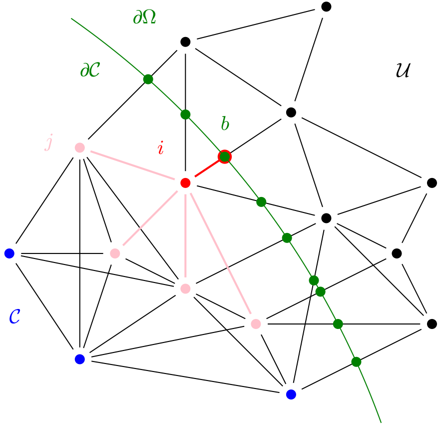


$$V_i \mathbb{V}_i^{\mathcal{C}} f = \sum_{j \in \mathring{\mathcal{C}}} \mathbf{A}_{i,j}^{\mathcal{U}} f_j + \sum_{b=(i,o) \in \partial\mathcal{C}} \frac{1}{\alpha_b} \mathbf{A}_{i,o}^{\mathcal{U}} f_b$$



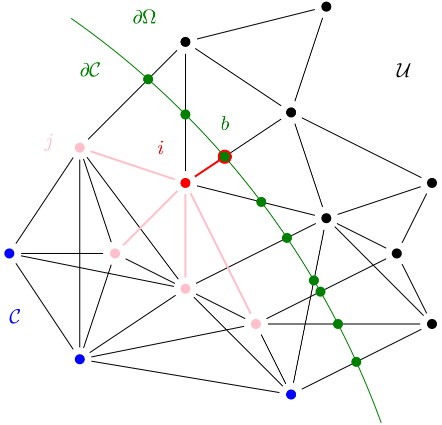
∂Ω
∂C
U
b
j
i
C

Embedding a point cloud in the geometry



$$H^1(\mathcal{C}) = \{u : \mathcal{C} \to \mathbb{R} \mid \forall b = (i, o) \in \partial\mathcal{C}, u_b = u_i + \mathbb{V}_i u \cdot (\mathbf{x}_b - \mathbf{x}_i)\}$$

Embedding a point cloud in the geometry



$$H^1_{0,D}(\mathcal{C}) = \{u : \mathcal{C} \to \mathbb{R} \mid \forall b = (i,o) \in \partial\mathcal{C} \backslash \partial\mathcal{C}_D, u_b = u_i + \nabla_i u \cdot (\mathbf{x}_b - \mathbf{x}_i)\}$$

$$\forall b = (i,o) \in \partial\mathcal{C}_D, \qquad u_b = 0$$

Interior nodes $\mathring{\mathcal{C}}$

- Volume integration V_i

Boundary nodes $\partial\mathcal{C}$

- Surface integration Γ_b

Separation of interior and boundary roles

Interior nodes $\mathring{\mathcal{C}}$

- Volume integration V_i
- Holds DOFs

Boundary nodes $\partial\mathcal{C}$

- Surface integration Γ_b
- Enforce BCs

Separation of interior and boundary roles

Interior nodes $\mathring{\mathcal{C}}$

- Volume integration V_i
- Holds DOFs
- Multiple boundary neighbors
  $\Leftrightarrow$ Cells in a mesh

Boundary nodes $\partial\mathcal{C}$

- Surface integration Γ_b
- Enforce BCs
- Single interior neighbor
  $\Leftrightarrow$ Faces of a cell

Final discrete weak formulation

Discrete weak formulation

Find $u : \mathcal{C} \to \mathbb{R}$ such that :

$$\begin{cases} \displaystyle\int_{\mathcal{C}} \nabla u \cdot \nabla v = \int_{\mathcal{C}} sv + \int_{\partial \mathcal{C}_N} gv \qquad \forall v \in H^1_{0,D}(\mathcal{C}) \\ u - u_0 \in H^1_{0,D}(\mathcal{C}) \end{cases}$$

CentraleSupélec get it right®

Exact linear solution ?

Find $u : \mathcal{C} \to \mathbb{R}$ such that :

$$\int_{\mathcal{C}} \nabla \mathbf{x} \cdot \nabla v = \int_{\partial \mathcal{C}_N} v \mathbf{n} \qquad \forall v \in H^1_{0,D}(\mathcal{C})$$

Exact linear solution ?

Find $u : \mathcal{C} \to \mathbb{R}$ such that :

$$\oint_{\mathcal{C}} \mathbb{\nabla}\mathbf{x} \cdot \mathbb{\nabla} v = \oint_{\partial\mathcal{C}_N} v\mathbf{n} \quad \forall v \in H^1_{0,D}(\mathcal{C})$$

Two necessary conditions :

- $\mathbb{\nabla}\mathbf{x} = \mathrm{I}_d$

  $\Leftrightarrow \mathbb{\nabla}$ is first order consistent

Exact linear solution ?

Find $u : \mathcal{C} \to \mathbb{R}$ such that :

$$\oint_{\mathcal{C}} \mathbb{\nabla}\mathbf{x} \cdot \mathbb{\nabla} v = \oint_{\partial\mathcal{C}_N} v\mathbf{n} \quad \forall v \in H^1_{0,D}(\mathcal{C})$$

Two necessary conditions :

- $\mathbb{\nabla}\mathbf{x} = \mathrm{I}_d$

  $\Leftrightarrow \mathbb{\nabla}$ is first order consistent (easy)

Exact linear solution ?

Find $u : \mathcal{C} \to \mathbb{R}$ such that :

$$\oint_{\mathcal{C}} \nabla\mathbf{x} \cdot \nabla v = \oint_{\partial\mathcal{C}_N} v\mathbf{n} \quad \forall v \in H^1_{0,D}(\mathcal{C})$$

Two necessary conditions :

- $\nabla\mathbf{x} = \mathrm{I}_d$

  $\Leftrightarrow$ ∇ is first order consistent (easy)

- $\oint_{\mathcal{C}} \nabla v = \oint_{\partial\mathcal{C}} v\mathbf{n} \quad \forall v : \mathcal{C} \to \mathbb{R}$

  $\Leftrightarrow$ Discrete version of Stokes' formula

  $\Leftrightarrow$ Compatibility between $\oint_{\mathcal{C}}$, $\oint_{\partial\mathcal{C}}$ and ∇.

Compatibility conditions in coordinates

In the interior

$$\forall i \in \mathring{\mathcal{C}}, \quad \sum_{j \in \mathring{\mathcal{C}}} \mathbf{A}_{j,i} = \mathbf{0}$$

$\Leftrightarrow$ Closedness of interior "dual cells"

CentraleSupélec get it right®

In the interior

$$\forall i \in \mathring{\mathcal{C}}, \quad \sum_{j \in \mathring{\mathcal{C}}} \mathbf{A}_{j,i} = \mathbf{0}$$

$\Leftrightarrow$ Closedness of interior "dual cells"

Identical to the boundary fitted case away from $\partial\mathcal{C}$

In the interior

$$\forall i \in \mathring{\mathcal{C}}, \quad \sum_{j \in \mathring{\mathcal{C}}} \mathbf{A}_{j,i} = \mathbf{0}$$

$\Leftrightarrow$ Closedness of interior "dual cells"

Identical to the boundary fitted case away from $\partial\mathcal{C}$

On the boundary

$$\forall b = (i, o) \in \partial\mathcal{C}, \quad \mathbf{A}_{i,b} = \Gamma_b \mathbf{n}_b$$

$\Leftrightarrow$ Gradient coefficients are vector boundary surface areas.

Corrected first order consistent gradient

- Necessary form :

$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j} (f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

Corrected first order consistent gradient

- Necessary form :

$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j} (f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

On the boundary only

- $\boldsymbol{\lambda}_{i,j} = \mathbf{0}$ if $\mathcal{N}(i) \cap \partial\mathcal{C} = \varnothing$

Correction for compatibility

Corrected first order consistent gradient

- Necessary form :

$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j} (f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

On the boundary only

- $\boldsymbol{\lambda}_{i,j} = \mathbf{0}$ if $\mathcal{N}(i) \cap \partial\mathcal{C} = \varnothing$
- Solve compatibility equations for $\boldsymbol{\lambda}_{i,j}$

Corrected first order consistent gradient

- Necessary form :

$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j}(f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

On the boundary only

- $\boldsymbol{\lambda}_{i,j} = \mathbf{0}$ if $\mathcal{N}(i) \cap \partial\mathcal{C} = \varnothing$
- Solve compatibility equations for $\boldsymbol{\lambda}_{i,j}$
- Sparse linear system

Corrected first order consistent gradient

- Necessary form :

$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j} (f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

On the boundary only

- $\boldsymbol{\lambda}_{i,j} = \mathbf{0}$ if $\mathcal{N}(i) \cap \partial\mathcal{C} = \varnothing$
- Solve compatibility equations for $\boldsymbol{\lambda}_{i,j}$
- Sparse linear system
- Size of system $\propto \#(\partial\mathcal{C})$

CentraleSupélec get it right®

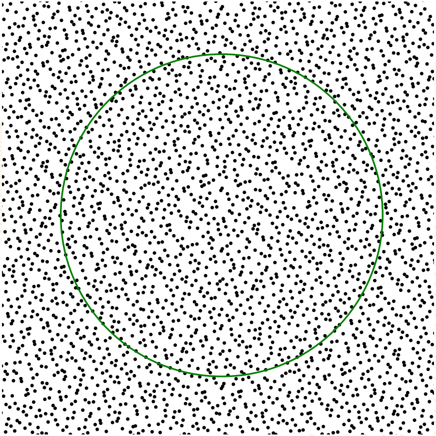
Corrected first order consistent gradient

- Necessary form :

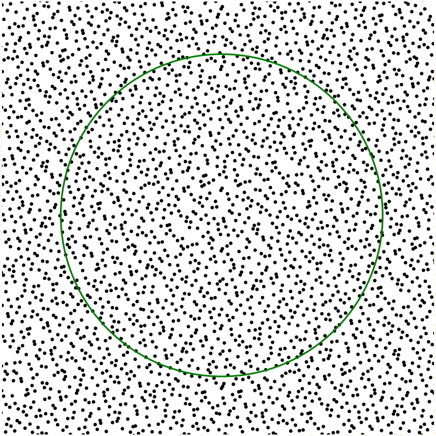
$$\widetilde{\nabla}_i f = \nabla_i f + \sum_{j \in \mathcal{C}} \boldsymbol{\lambda}_{i,j} (f_j - f_i - (\mathbf{x}_j - \mathbf{x}_i) \cdot \nabla_i f)$$

On the boundary only

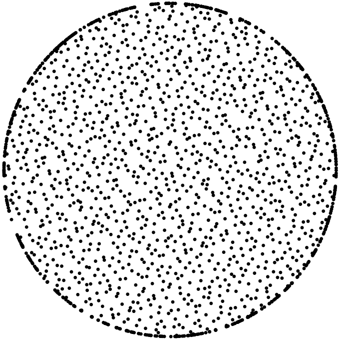
- $\boldsymbol{\lambda}_{i,j} = \mathbf{0}$ if $\mathcal{N}(i) \cap \partial\mathcal{C} = \varnothing$
- Solve compatibility equations for $\boldsymbol{\lambda}_{i,j}$
- Sparse linear system
- Size of system $\propto \#(\partial\mathcal{C})$
- Ill-conditioned : $\kappa \propto h^4$



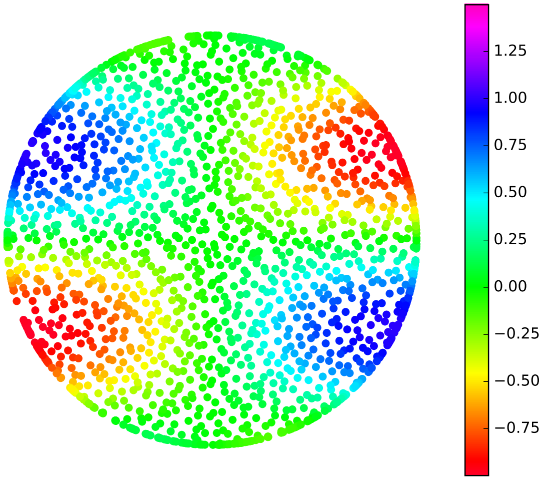
Initial cloud $\mathcal{U}$:
Halton distribition



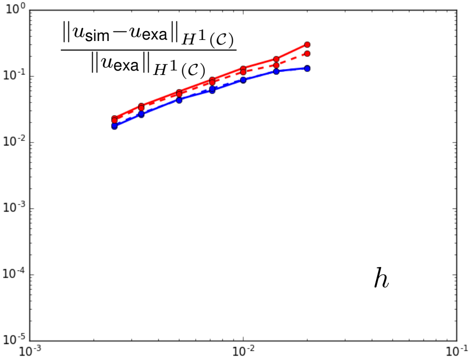
Initial cloud $\mathcal{U}$:
Halton distribition



Trimmed cloud $\mathcal{C}$



$$u_{\text{exact}} = \sin(k_x x)\sin(k_y y)$$



SPH-like volumes

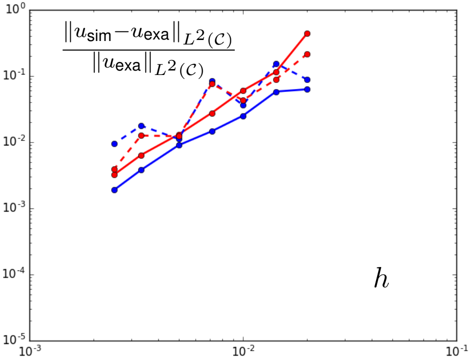
Uniform volumes

Plain curve :
Full Dirichlet

Dashed curve :
Neumann + Dirichlet

linear fit : 0.97 - 1.19

$\Rightarrow$ First order convergence



SPH-like volumes

Uniform volumes

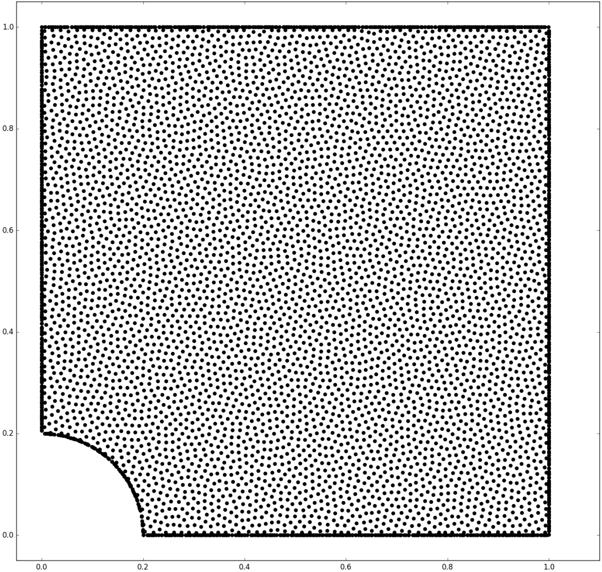
Plain curve :
Full Dirichlet

Dashed curve :
Neumann + Dirichlet

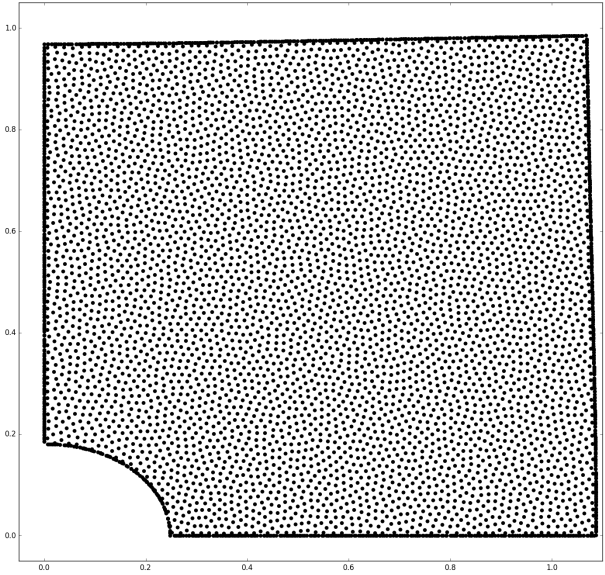
linear fit :

Dirichlet : 1.72 - 2.23
Neumann : 1.24 - 1.73

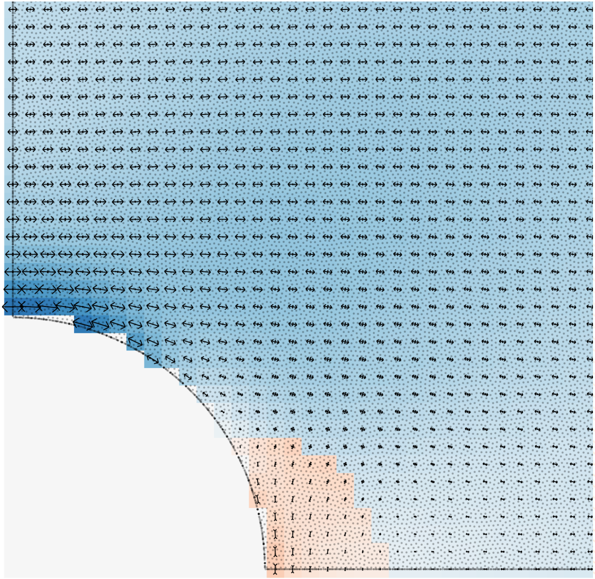
Elasticity simulations : stress concentration



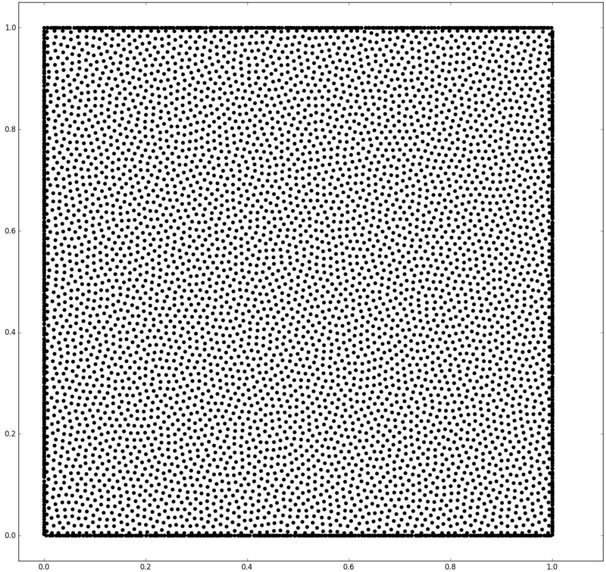
Elasticity simulations : stress concentration



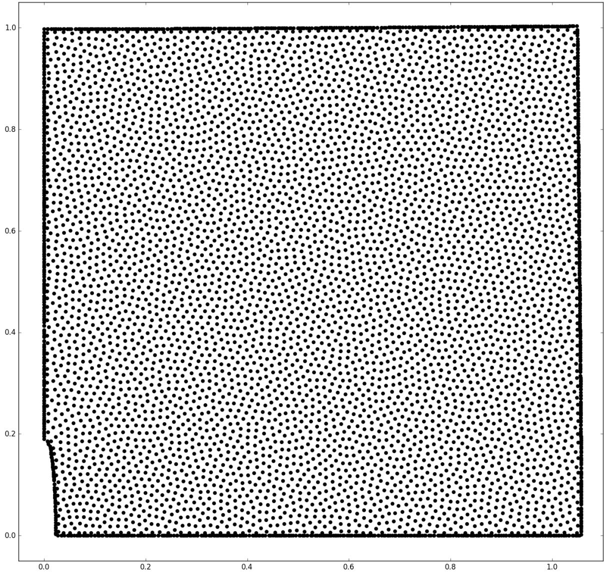
Elasticity simulations : stress concentration

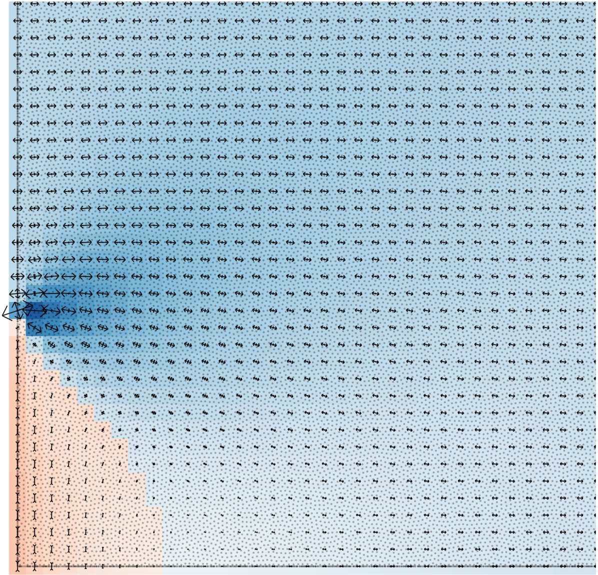


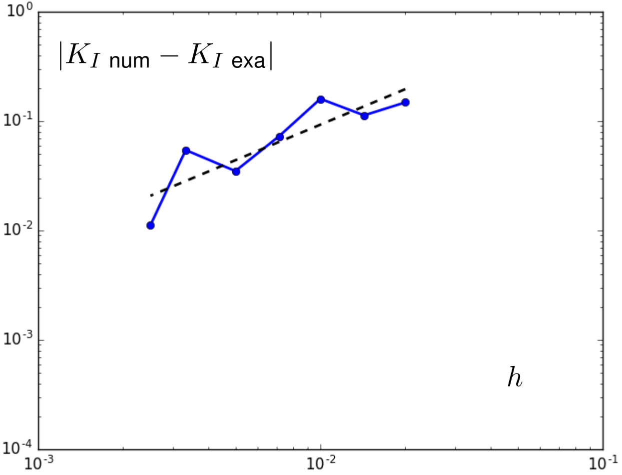
Stress intensity factor at crack



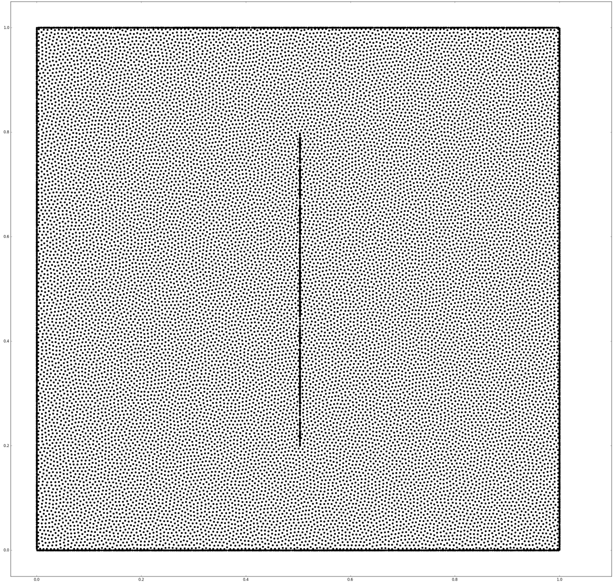
Stress intensity factor at crack



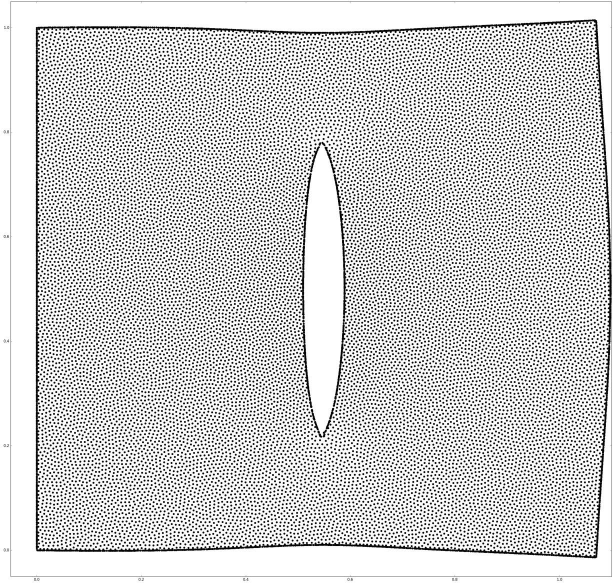



$|K_{I\ \mathrm{num}} - K_{I\ \mathrm{exa}}|$
h
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

Inner boundaries



Inner boundaries



CentraleSupélec get it right®

Summary

- Proposition of an immersed meshless method
- Good H^1 behavior
- Allows the computation of stress intensity factors

Summary

- Proposition of an immersed meshless method
- Good H^1 behavior
- Allows the computation of stress intensity factors

Ongoing and future work

- Investigate stability and L^2 behavior
- Simulate crack propagation

Thanks for your attention !

gabriel.fougeron@esi-group.com guillaume.pierrot@esi-group.com

denis.aubry@ecp.fr